is proved the transcendence of e. Closely following Hermite came the same proof for π by Lindemann in a dissertation *Ueber die Zahl* π (*Math. Ann.* 20, 1882. See also the proceedings of the Berlin and Paris academies). With this the matter was now for the first time settled, nevertheless the treatment given by Hermite and Lindemann is very complicated.

The first simplification was given by Weierstrass in the *Berliner Berichte* in 1885. The above mentioned works Bachman embodied in his text-book, *Vorlesungen ueber die Natur der Irrationalzahlen*, 1892.

The spring of 1893 brought, however, new and very important simplifications. In the first rank should be named the developments of Hilbert in the *Göttinger Nachrichten.* Hilbert's proof is not wholly elementary; it contains still a remnant of Hermite's course of reasoning in the integral

$$\int_0^\infty z^\rho e^{-z} dz = \rho\,!.$$

But Herwitz and Gordan soon after showed that this transcendental part might be eliminated. (*Göttinger Nachrichten* and *Comptes Rendus* respectively; all three dissertations are reproduced in the *Math. Annalen*, Bd. 43, either literally or somewhat extended). So the matter has now become so elementary that it is generally available.

INTRODUCTION TO SUBSTITUTION GROUPS.

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[Continued from November Number.]

THE CONSTRUCTION OF NON-PRIMITIVE GROUPS WITH TWO SYSTEMS OF NON-PRIMITIVITY.

Let the degree of the required non-primitive group be $2n$, and consider the $(n\,!)^2$ substitutions

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all } a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n$$

and also the group of order $2(n\,!)^2$

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n).$$

The latter is clearly a non-primitive group of degree $2n$ and the former are the substitutions of this group which interchange the systems. It is easily seen

that G_1 can have no larger value than it has in the above non-primitive group, and that every G_1 for other non-primitive groups may be regarded as a subgroup of this G_1. From this it follows that the first set of substitutions includes all the substitutions which can be used with any G_1 to form a non-primitive group, for if there were such a substitution s_x which is not in the first set then we would obtain more than $(n!)^2$ different substitutions which transform

$$(a_1 a_2 \ldots\ldots a_n)\mathrm{all}(b_1 b_2 \ldots\ldots b_n)\mathrm{all}$$

into itself without interchanging the systems by multiplying one substitution of this set into the entire set increased by s_x. Hence all the substitutions which can be used to interchange the systems are found in the first set. In a similar way we can show that the number of the snbstitutions which interchange the systems must *always* be equal to the order of G_1. Hence if in any non-primitive group we represent the substitution which interchange the systems by G_2 and the non-primitive group by G we have

$$G = G_1 + G_2$$

where G_1 and G_2 contain the same number of substitutions and G is a subgroup of

$$(a_1 a_2 \ldots\ldots a_n)\mathrm{all}(b_1 b_2 \ldots\ldots b_n)\mathrm{all}(a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n).$$

Suppose any G_1 constructed by combining a transitive* subgroup of $(a_1 a_2 \ldots\ldots a_n)$all with a conjugate subgroup of $(b_1 b_2 \ldots\ldots b_n)$all and suppose s_γ to have the following properties:

(1) its square is found in G_1;

(2) it transforms G_1 into itself;

(3) it interchanges the systems of G_1.

Then will all of the substitutions

$$G_1 s_\gamma = G_2$$

*We shall henceforth assume that the systems of non-primitivity are the transitive constituents of G_1. We proved above that this can always be done but we did not prove that it is possible to regard intransitive constituents of G_1 as systems. That this may be done is proved by the following instance in which

$$G_1 = (ab.cd.ef.gh.ij.kl)$$

and the systems are either a,b; c,d; e,f; g,h; i,j; and k,l or a,b,c,d; e,f,g,h; i,j,k,l. Letting the letters A, B, etc., stand for the first systems and A', B', C' for the second we may write the group as follows:

$$\left.\begin{array}{l} 1 \\ ab.cd.ef.gh.ij.kl \end{array}\right\} 1$$

$$\left.\begin{array}{l} aei.bfj.cgk.dhl \\ afibej.chkdgl \end{array}\right\} \begin{array}{l} ABC.DEF \\ A'B'C' \end{array}$$

$$\left.\begin{array}{l} aie.bjf.ckg.dlh \\ ajebif.clgdkh \end{array}\right\} \begin{array}{l} ACB.DFE \\ A'C'B' \end{array}$$

$$\left.\begin{array}{l} ac.bd.ek.fl.gi.hj \\ ad.bc.el.fk.gj.hi \end{array}\right\} \begin{array}{l} AD.BF.CE \\ B'C' \end{array}$$

$$\left.\begin{array}{l} ag.bh.ce.df.ik.jl \\ ah.bg.cf.de.il.jk \end{array}\right\} \begin{array}{l} AE.BD.CF \\ A'B' \end{array}$$

$$\left.\begin{array}{l} ak.bl.ci.dj.eg.fh \\ al.bk.cj.di.eh.fg \end{array}\right\} \begin{array}{l} AF.BE.CD \\ A'C' \end{array}$$

If we consider the six systems they are the transitive constituents of G_1 but if we consider only the three systems they are intransitive constituents.

have these properties and it can be easily seen that G_1+G_2 constitute a non-primitive group. Hence it follows that it is only necessary to find one substitution which possesses the three properties named above in order to obtain a G_2 corresponding to a given G_1.

To fix these ideas we proceed to find all the non-primitive groups whose degree does not exceed six. Since n must be the degree of some group it follows that $2n$ cannot be less than four.

Non-Primitive Groups of Degree Four.

G_1 must be either $(ac.bd)$ or $(ac)(bd)$. G_2 is found in $(ac)(bd)ab.cd=ab.cd$, $abcd$, $adcb$, $ad.bc$. If $G_1=(ac.bd)$ we see at once that $ab.cd$ and $abcd$ satisfy the three required conditions. In the first case $G_2=ab.cd$, $ad.bc$ and in the second case it equals $abcd$, $adcb$. Hence the given G_1 leads to the following two non-primitive groups of degree and order four : (A transitive group is called *regular* when its degree is equal to its order.)

$$(abcd)_4,\ (abcd)\text{cyc}.$$

If $G_1=(ac)(bd)$ we see again directly that $ab.cd$ satisfies the three required conditions, as we found in the general case. The corresponding G_2 includes all the possible substitutions. We obtain therefore only one non-primitive group with this G_1, viz :

$$(abcd)_8$$

Hence there are *three* non-primitive groups of degree four. The other two transitive groups of degree four are multiply transitive and therefore primitive.

Non-Primitive Groups of Degree Six with Two Systems of Non-Primitivity.

G_1 must be one of the following five groups :

$$(abc)\text{all}(def)\text{all},\ \{(abc)\text{all}(def)\text{all}\}\text{pos},\ (abc.def)\text{all}$$
$$(abc)\text{cyc}(def)\text{cyc},\ (abc.def)\text{cyc}.$$

G_2 is found in

$$(abc)\text{all}(def)\text{all}\ ad.be.cf.$$

(a) If $G_1=(abc)\text{all}(def)\text{all}$, G_2 will include all the possible substitutions and we obtain one group of order 72, viz :

(1) $$(abc)\text{all}(def)\text{all}(ad.be.cf).$$

(b) If $G_1=\{(abc)\text{all}(def)\text{all}\}\text{pos}$ the two substitutions $ad.be.cf$ and $aebd.cf$ satisfy the three conditions and we thus obtain one G_2 which contains only negative substitutions and another which contains only positive substitutions. The two resulting groups are

(2) $\{(abc)\text{all}(def)\text{all}\}\,\text{pos}(ad.be.cf)=(abcdef)_{36_3}$ *

(3) $\{(abc)\text{all}(def)\text{all}\}\,\text{pos}(aebd.cf)=(abcdef)_{36}$.

(*c*) If $G_1=(abc.def)\text{all}\ ad.be.cf$ satisfies the three necessary conditions. We thus obtain

(4) $$(abc.def)\text{all}(ad.be.cf)=(abcdef)_{12}.$$

No substitutions except those in the above G_2 can transform $(abc.def)$all into itself and interchange the systems, because no two substitutions of (abc)all transform (abc)all in the same way. Hence there is only one G_2 for the given G_1.

(*d*) If $G_1=(abc)\text{cyc}(def)\text{cyc}$, then the square of only half of the substitutions in which G_2 is found are contained in this G_1. Hence two G_2's are possible, viz:

$$G_1\ ad.be.cf \text{ and } G_1\ ab.de.ad.be.cf=G_1\ ae.bd.cf:$$

ab transforms G_1 into itself and one of these G_2's into the other so that there is really only the following non-primitive group with the given G_1†:

(5) $$(abc)\text{cyc}(def)\text{cyc}(ad.be.cf).$$

(*e*) Finally if $G_1=(abc.def)\text{cyc}$ we obtain two G_2's and hence the following groups:

(6) $$(abc.def)\text{cyc}(ad.be.cf)=(abcdef)\text{cyc}$$

(7) $$(abc.def)\text{cyc}(ae.bd.cf)=(abcdef)_6$$

The first one of these two will be found in three conjugate forms if we use all the possible G_2's. We have now examined all the possible G_1's and found seven non-primitive groups of degree six which contain two systems of non-primitivity.

[To be Continued.]

*This group is not found in Professor Cayley's list, *Quarterly Journal of Mathematics*, Vol. 25. pp. 71—79. It is found in Professor Cole's supplementary list, *Bulletin of the New York Mathematical Society*, May, 1893.

†It has been proved that whenever G_1 is the product of two groups then there is really only one G_2 for the given G_1. We shall give a proof of this theorem later.